Then we can rewrite the *Yule-Walker* equations, Equation 5.65, in matrix form as follows,

$$_l\mathbf{R}\ _l\mathbf{a} = {_l\mathbf{r}} \tag{5.70}$$

where $_l\mathbf{a} : \mathscr{R}^1 \mapsto \mathscr{R}^Q$ is the vector of the *all-pole* parameters for frame l.

Therefore, the least squares solution to the *all-pole* estimate is given by,

$$_l\mathbf{a} = {_l\mathbf{R}^{-1}}\ _l\mathbf{r} \tag{5.71}$$

Note the structure of $_l\mathbf{R}$ in the definition of Equation 5.68. $_l\mathbf{R}$ is a *Toeplitz* matrix which shows up in many solutions to difference equations, especially in control systems and signal processing. Its structure makes it quite simple to solve for $_l\mathbf{a}$ using the *Levinson-Durbin* algorithm which was introduced by *Levinson* [42] and then modified by *Durbin* [18, 19, 8, 60]. There is also another algorithm called *Schür recursion* [66], which is more efficient for parallel implementations.

From the spectral perspective, due to Parseval's theorem (see Section 24.9.7), element q of $_l\mathbf{r}$, denoted by $_l\mathbf{r}|_q$ is as follows [46],

$$\begin{aligned} _lr(j) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} {_l\mathscr{P}_d^\circ}(\omega)\ \cos(j\omega)d\omega \\ &= \sum_{n=-\infty}^{\infty} h_n h_{(n+j)} \end{aligned} \tag{5.72}$$

Note that in the spectral form of Equation 5.72, the angular frequency, ω, is treated as a continuous variable. It is possible to discretize the frequency as we did in the DFT method by Equation 24.461 so that we obtain the discretized version of Equation 5.72, namely,

$$\begin{aligned} _lr(j) &= \sum_{0}^{N-1} {_l\mathscr{P}_d^\circ}(\omega)\ \cos(2\pi kj) \\ &= \sum_{n=0}^{N-1-Q} {_lh_n}\ {_lh_{n+j}} \end{aligned} \tag{5.73}$$

In the discretized version,

$$E = \frac{G^2}{N}\sum_{n=0}^{N-1}\frac{\mathscr{P}(\omega_k)}{\hat{\mathscr{P}}(\omega_k)} \tag{5.74}$$

Equation 5.74 means that only discrete values of the frequency are contributing to the error being computed. This means that the minimum error is only valid for the discrete frequencies within the range of ω_k. This re-iterates the fact that if there are higher frequencies present, they will not be modeled. Note that one other possible method for computation of the *all-pole* estimate is to use a discrete cosine transform through an FFT as apparent by Equation 5.74 See [46] for a complete treatment of the spectral path of the solution of the Linear Predictive Coefficients.

Let us return to the Yule-walker equations. As we mentioned, up to the windowing step, the LPC method is identical to the *direct method* of Section 5.3. In the next step, we will use the short-time autocorrelation (Equation 5.67) for the N-sample frame and solve Equation 5.71 to compute the LPC coefficients and then follow on to compute the LPCC features.

5.4.2 LPC Computation

At this point we have arrived at Equation 5.71 which should be solved for every frame of the signal. As we noted, the Toeplitz structure of the autocorrelation matrix, ${}_l\mathbf{R}$, allows us to use the efficient Levinson-Durbin method to solve the Yule-walker Equations (Equation 5.70) directly, without having to compute ${}_l\mathbf{R}^{-1}$, for the, so called, Linear Predictive Coding (LPC) coefficients, ${}_la_q, q \in \{1, 2, \cdots, Q\}$ and $l \in \{0, 1, \cdots, L-1\}$.

The following pseudo-code represents the steps of the Levinson-Durbin method as stated by *Rabiner and Juang* [60] (some typographical errors which existed in [60] have been corrected here),

Initialize E:

$$E^{(0)} = {}_lr(0) \tag{5.75}$$

for (q = 1 to Q),

1.

$$ {}_l\kappa_q = \frac{{}_lr(q) - \sum_{j=1}^{q-1} \alpha_j^{(q-1)} {}_lr(q-j)}{E^{(q-1)}} \tag{5.76}$$

2.

$$\alpha_q^{(q)} = {}_l\kappa_q \tag{5.77}$$

3. for (j = 1 to Q),

$$\alpha_j^{(q)} = \alpha_j^{(q-1)} - {}_l\kappa_q \alpha_{q-j}^{(q-1)} \tag{5.78}$$

endfor

$$E^{(q)} = (1 - {}_l\kappa_q^2) E^{(q-1)} \tag{5.79}$$